M.U.
M.Sc. 95

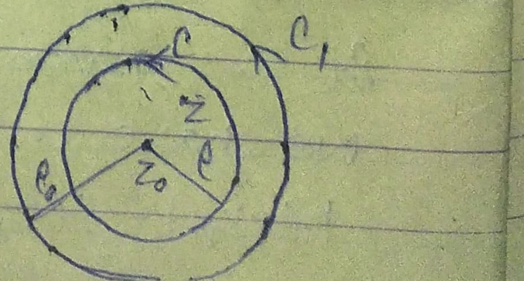
Q No → State and Prove Taylor's theorem.

Ans → Statement:- If $f(z)$ is analytic in a circular domain D with centre z_0 ~~then and~~ ~~then~~ for every $z \in D$.

$$f(z) = f(z_0) + \frac{(z-z_0)}{\lfloor 1} f'(z_0) + \frac{(z-z_0)^2}{\lfloor 2} f''(z_0) + \cdots + \frac{(z-z_0)^n}{\lfloor n} f^n(z_0) + \cdots$$

$$= f(z_0) + \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{\lfloor n} f^n(z_0)$$

Proof: Let z be any point within the circle C_1 with the centre z_0 and radius ρ_0. Let $|z-z_0| = r$ and let C be the circle with the centre z_0 and radius ρ such that $r < \rho < \rho_0$.



Now, by Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi_1)}{\xi_1 - z}\, d\xi_1 \quad \text{—} \ (1)$$

$$\frac{1}{\xi_1 - z} = \frac{1}{(\xi_1 - z_0)(z - z_0)} = \frac{1}{(\xi_1 - z_0)\left\{1 - \frac{z - z_0}{\xi_1 - z_0}\right\}}$$

$$= \frac{1}{\xi_1 - z_0}\left(1 - \frac{z - z_0}{\xi_1 - z_0}\right)^{-1}$$

$$= \frac{1}{\xi_1 - z_0}\left\{1 + \frac{z - z_0}{\xi_1 - z_0} + \left(\frac{z - z_0}{\xi_1 - z_0}\right)^2 + \left(\frac{z - z_0}{\xi_1 - z_0}\right)^3 + \cdots \right.$$

$$\left. \cdots + \left(\frac{z - z_0}{\xi_1 - z_0}\right)^{n-1} + \frac{(z - z_0)^n}{(\xi_1 - z_0)^n}\frac{1}{1 - \frac{z - z_0}{\xi_1 - z_0}}\right\}$$

$$= \frac{1}{\xi_1 - z_0} + \frac{z - z_0}{(\xi_1 - z_0)^2} + \frac{(z - z_0)^2}{(\xi_1 - z_0)^3} + \cdots + \frac{(z - z_0)^{n-1}}{(\xi_1 - z_0)^n} + \frac{(z - z_0)^n}{(\xi_1 - z_0)^n}\frac{f}{\xi_1 - z}$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(\xi_1)}{\xi_1 - z}\, d\xi_1 = \frac{1}{2\pi i}\int_C \frac{f(\xi_1)}{\xi_1 - z_0}\, d\xi_1 + \frac{(z - a)}{2\pi i}\int_C \frac{f(\xi_1)}{(\xi_1 - z_0)^2}\, d\xi_1$$

$$+ \cdots + \frac{(z - z_0)^{n-1}}{2\pi i}\int_C \frac{f(\xi_1)}{(\xi_1 - z_0)^n}\, d\xi_1 + \frac{1}{2\pi i}\int_C \frac{(z - z_0)^n f(\xi_1)}{(\xi_1 - z_0)^n(\xi_1 - z)}\, d\xi_1$$

$$= f(z_0) + \frac{(z - z_0)}{\lfloor 1}f'(z_0) + \frac{(z - z_0)^2}{\lfloor 2}f''(z_0) + \cdots + \frac{(z - z_0)^{n-1}}{\lfloor n-1}$$

$$f^{n-1}(z_0) + R_n \quad \text{—} \ (2)$$

Where, $R_n = \frac{1}{2\pi i}\int_C \frac{(z - z_0)^n f \xi_1}{(\xi_1 - z_0)^n(\xi_1 - z)}\, d\xi_1$

The Prove will be Complete, It be Can Show that $R_n \to 0$ as $n \to \infty$, We have

$$|\xi_1 - z| = |(\xi_1 - z_0) - (z - z_0)| \geq |\xi_1 - z_0| - |z - z_0| = \rho - r$$

Hence, $|R_n| = \left| \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{(z-z_0)^n f(q)}{(q-z_0)^n (q-z)} dq \right|$

$$\leq \frac{|z-z_0|^n}{|2\pi i|} \int_C \frac{|f(q)|\,|dq|}{|q-z_0|^n (q-z)} \leq \frac{r^n \cdot M}{2\pi(\rho-r)\rho^n}$$

$$\therefore |R_n| \leq \frac{M}{2\pi(\rho-r)} \cdot \left(\frac{r}{\rho}\right)^n 2\pi\rho$$

$$= \frac{M\rho}{\rho-r}\left(\frac{r}{\rho}\right)^n \to 0 \text{ as } n \to \infty \text{ since } \frac{r}{\rho} <$$

$$\therefore f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{\lfloor n} f^n(z_0)$$